Stochastic Response of Nonlinear Structures with Parameter Random Fluctuations

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The random response of a nonlinear structural system is examined when its parameters are experiencing random fluctuations with time. The treatment is based on the recent developments in the mathematical theory of stochastic differential equations. These include the Ito stochastic calculus and the Fokker-Planck equation approach to derive a general differential equation that describes the evolution of the statistical moments of the response coordinates. The differential equation is found to constitute an infinite coupled set of differential equations that are closed via two different closure schemes. The system response is determined in the neighborhood of internal resonance condition and for various random intensities of the system parameters. It is found that the random modal interaction is governed mainly by the internal resonance ratio and the stiffness fluctuation intensity. The effect of the random damping fluctuation on the system response is found to be very small compared to the stiffness fluctuation effect.

I. Introduction

THE dynamic behavior of lightweight structures is of main concern to aeronautical engineers involved in the design and reliability of aerospace structures. These structures are usually made up of composite materials that are nonhomogeneous and exhibit fluctuations in their dynamic properties. The fluctuations of these properties are random in nature and thus result in random eigenvalues and responses. Depending on the analytical modeling of such structures, the interaction between aerodynamic, inertia, and elastic forces may give rise to a number of aeroelastic phenomena. For example, classical flutter can occur due to a linear interaction of these three forces. Classical flutter may also involve the coupling of two or more degrees of freedom. However, the linear mathematical modeling fails to predict a number of observed dynamic characteristics such as amplitude jump, limit cycles, parametric instability, internal resonance, multiple solutions, and saturation phenomena. These complex characteristics owe their origin to the inherent nonlinearity of the structure.

The amplitude jump, limit cycles, and parametric instability are common features of nonlinear single- and multi-degree-of-freedom systems. Parametric instability[1] takes place when the external excitation appears as a coefficient in the homogeneous part of the equation of motion. It occurs when the excitation frequency is twice the natural frequency of the system. Internal resonance[2] and saturation phenomena[3] may occur only in nonlinear dynamic systems with more than one degree of freedom. Internal resonance implies the existence of a linear relationship between the normal mode frequencies of the structure and results in a nonlinear interaction between the normal modes in a form of energy exchange. Under external excitation, the mode that is directly excited exhibits, in the beginning, the same features of the response of a linear single-degree-of-freedom system, and all other modes remain dormant. As the excitation amplitude reaches a certain critical level, the other modes become unstable, and the originally excited mode reaches an upper bound. This mode is said to be saturated, and the energy is then transferred to other modes. This type of modal interaction is referred to in the literature as autoparametric interaction,[4] since one mode acts as a parametric excitation to other modes. Barr and Done[5] conducted a ground resonance test and applied a sinusoidal excitation at one or more points to find out the conditions under which parametric and autoparametric instabilities could occur. The autoparametric resonance was found to take place when the directly excited mode frequency is twice the indirectly excited mode. Barr and Done[5] observed several combinations of normal mode interaction. For example, when the exciting mode was wing bending, the excited mode was found to be one of the following: 1) wing store pylon bending, 2) wing store pylon twisting, 3) engine pod mounting structure bending, or 4) engine mounting structure twisting.

In structural dynamics, the nonlinearity is represented in three different forms:[1,6] elastic, inertia, and damping nonlinearities. Elastic nonlinearity stems from nonlinear strain-displacement relations, which are inevitable. Inertia nonlinearity is derived, in a Lagrangian formulation, from kinetic energy. The equations of motion of a discrete mass dynamic system, with holonomic scleronomic constraints, in terms of the generalized coordinates q_i, are usually written in the form[7]

$$\sum_{j=1}^{n} m_{ij} q_j + \sum_{j=1}^{n} \sum_{\ell=1}^{n} [j\ell, i] q_i q_\ell + \frac{\partial V}{\partial q_i} = Q_i, \qquad i = 1, 2, \ldots, n$$

where V is potential energy, Q_i represents all nonconservative forces, and $[j\ell, i]$ is the Christoffel symbol of the first kind and is given by the expression

$$[j\ell, i] = \frac{1}{2}\left(\frac{\partial m_{ij}}{\partial q_\ell} + \frac{\partial m_{i\ell}}{\partial q_j} - \frac{\partial m_{j\ell}}{\partial q_i}\right)$$

The metric tensor m_{ij} and the Christoffel symbol are generally functions of the q_k, and for motion about the equilibrium configuration they can be expanded in a Taylor series about that state. Thus, from inertia sources, quadratic, cubic, and higher-power nonlinearities can arise.

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Other types of nonlinearities, such as distributed and concentrated nonlinearities, are encountered in aeroelastic flutter problems.[8] Distributed nonlinearity is induced by elastic deformations in riveted, screwed, and bolted connections, as well as the structural components themselves. Concentrated nonlinearity acts locally in control mechanisms or in the connecting parts between wing and external stores. This nonlinearity is caused by backlash in the linkage elements of the control system, dry friction in control cable and push rod ducts, kinematic limitation of the control surface deflection, and application of spring tab systems provided for relieving pilot operation. Breitbach[9] determined the flutter boundaries for three different configurations distinguished by different types of nonlinearities in the rudder and aileron control system of a sailplane. It was shown that the influence of hysteretic damping results in a considerable stabilizing effect and an increase in the flutter speed. Similar effects of nonlinearities due to friction and backlash were reported in Ref. 10. Peloubet et al.,[11] Reed et al.,[12] and Desmarais and Reed[13] examined the effect of control system nonlinearities, such as actuator force or deflection limits, on the performance of an active flutter suppression system. It was shown[12] that a nonlinear system that is stable with respect to small disturbances may be unstable with respect to large ones. Another important feature was that a store mounted on a pylon with low pitch stiffness can provide substantial increase in flutter speed and reduce the dependency of flutter on the mass and inertia of stores relative to that of still-mounted stores.

It is clear that, in mathematical modeling, the aeroelastician should consider various types of nonlinearities in order to understand the origin of any unusual structural behavior under various types of aerodynamic loadings. Under deterministic unsteady aerodynamic forces, these phenomena can be predicted by one of the standard perturbation techniques of nonlinear differential equations.[14,15] However, aerospace structures are usually subjected to turbulent airflow, and the analyst encounters aerodynamic loads that are random in nature. Furthermore, once the structure starts to vibrate, its parameters, such as damping and stiffness, experience random fluctuations with the passage of time. The dynamic analysis of these structures is not a simple task, and it requires an advanced background in probability theory and stochastic differential equations.

It is very important at this stage to distinguish between two different problems encountered in structural dynamics. These are the random response of dynamic systems to random parametric excitation[16] and the random response of structural systems whose parameters are random variables described in a probabilistic sense. In the former case, the system equations of motion are stochastic differential equations with coefficients that are random processes while, in the latter case, the equations of motion are differential equations with parameter uncertainties. The methods of treating dynamic systems under parametric random excitations are different from those used in solving differential equations with parameter uncertainties. Parametric random vibration is basically a combination of the theory of stochastic processes, stochastic differential equations, and applied dynamics. On the other hand, systems with parameter uncertainties (referred to in the literature as disordered systems)[17] involve random boundary-value problem and random field theory. Within the framework of the linear theory of random vibration of disordered systems, the engineer encounters problems of random eigenvalues, random eigenvectors, random responses, normal mode localization, optimum design, and reliability. These issues have recently been reviewed by the author in Ref. 18.

This paper deals with the random response of a nonlinear two-degree-of-freedom structural model when its damping and stiffness coefficients involve random time variation. The model consists of two coupled beams with tip concentrated masses as shown in Fig. 1. The deterministic response of this model to harmonic support motion has been examined by Barr and Ashworth,[19] and Haddow et al.[20] Their studies showed that the system exhibits a number of nonlinear response phenomena under conditions of internal resonance, high excitation level, and low damping ratios. The authors of the present paper have recently determined the random response of this system to random support motion when its parameters are time-independent.[21] In fact, as the structure oscillates, the damping and stiffness coefficients may experience random time variations. The random variations of the structural parameters and the random support excitation will be assumed Gaussian-independent wide-band processes. The Fokker-Planck equation approach will be used to generate a general first-order differential equation for the statistical moments of the response coordinates. In view of the system nonlinearity, the response processes will be non-Gaussian-distributed, and the moment equations will form an infinite coupled set of moment equations, which will be closed via two independent closure schemes referred to as Gaussian and non-Gaussian closure schemes.[16] These closure schemes are based on the semi-invariant properties of the response processes. The Gaussian closure scheme is only valid if the system is linear with time-invariant coefficients and is subjected to Gaussian excitation. The application of Gaussian closure to nonlinear systems is analogous to the linearization solutions of deterministic nonlinear differential equations. The non-Gaussian closure scheme is more accurate since it takes into account the effect of the system nonlinearity on the response probability density function. More details of closure schemes may be found in a recent research monograph[16] by the first author. The results will be compared with the response statistical moments of the same system when its coefficients are constants.

II. Theoretical Analysis

Equations of Motion and Response Markov Vector

Figure 1 shows a schematic diagram of an analytical model of a nonlinear aeroelastic structural system, which represents a wing with external store. It consists of two coupled beams with tip masses m_1 and m_2. The present study will examine the nonlinear random interaction between the first two normal modes under random support motion $\xi_0(t)$ when the dynamic properties of the system experience random fluctuations. The mathematical modeling of the system was derived in Ref. 22. In terms of the nondimensional normal coordinates Y_1 and Y_2, the system equations of motion are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} Y_1'' \\ Y_2'' \end{Bmatrix} + \begin{bmatrix} 2\zeta_1[1+\xi_{c1}(\tau)] & 0 \\ 0 & 2r\zeta_2[1+\xi_{c2}(\tau)] \end{bmatrix} \begin{Bmatrix} Y_1' \\ Y_2' \end{Bmatrix} + \begin{bmatrix} 1+\xi_{k1}(\tau) & 0 \\ 0 & r^2[1+\xi_{k2}(\tau)] \end{bmatrix} \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} = -\xi_0''(\tau) \begin{Bmatrix} a_1 \\ b_1 \end{Bmatrix} - \epsilon\xi_0''(\tau) \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} + \begin{Bmatrix} \psi_1(Y, Y', Y'') \\ \psi_2(Y, Y', Y'') \end{Bmatrix} \tag{1}$$

where $\{Y_1, Y_2\} = \{y_1, y_2\}/q_1^0$, and q_1^0 is the response root-mean-square of the system when the vertical beam is locked and the horizontal beam behaves as a single degree of freedom with end mass $m_1 + m_2$.

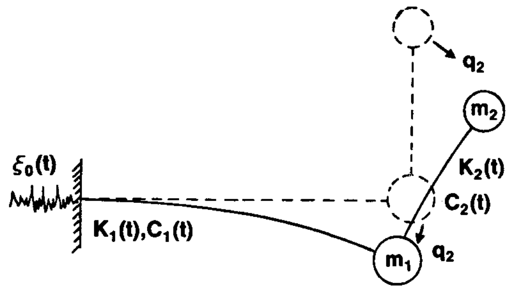

Fig. 1 Schematic diagram of a structural model with random parameters under random support motion.

The normal coordinates y are related to the generalized coordinates q through the transformation

$$\{q\} = [R]\{y\} \tag{2}$$

where $[R]$ is the system modal matrix that is defined in Ref. 21. The small parameter $\epsilon = q_1^0/\ell_1$ is the nonlinear coupling parameter, $r = \omega_2/\omega_1$ is the frequency ratio where ω_1 and ω_2 are the normal mode frequencies (in the present paper it is considered that $\omega_2 < \omega_1$). A prime denotes differentiation with respect to the nondimensional time parameter $\tau = \omega_1 t$.

The nonlinear functions $\psi_i(Y, Y', Y'')$ are given by the expressions

$$\psi_1(Y, Y', Y'') = a_4 Y_1 Y_1'' + a_5 Y_1 Y_2'' + a_6 Y_2 Y_1'' + a_7 Y_2 Y_2'' + a_8 Y_1'^2 + a_9 Y_1' Y_2' + a_{10} Y_2'^2$$

$$\psi_2(Y, Y', Y'') = b_4 Y_1 Y_1'' + b_5 Y_1 Y_2'' + b_6 Y_2 Y_1'' + b_7 Y_2 Y_2'' + b_8 Y_1'^2 + b_9 Y_1' Y_2' + b_{10} Y_2'^2$$

The coefficients a_i and b_i depend on the system parameters and are defined in Ref. 22. These functions involve quadratic nonlinearities of the inertia type. They include autoparametric coupling terms such as $Y_1 Y_2''$ in which the acceleration Y_2'' of the second mode acts as a parametric excitation to the first mode. The first expression on the right-hand sides of Eqs. (1) represents the nonhomogeneous part of the excitation, while the second expression is the parametric action of the excitation that couples the two modes parametrically. $\xi_{ci}(\tau)$ and $\xi_{ki}(\tau)$ represent the random fluctuations in the damping and stiffness terms, respectively. These functions and the support acceleration are assumed to be Gaussian wide-band random processes with zero means. The spectral densities of these processes are assumed to cover a frequency band that includes the first two normal mode frequencies and to be well below any other higher normal mode frequency. In the limiting case, as the correlation time of $\xi_i(\tau)$ becomes very small compared with any characteristic period of the system, the response coordinates approach a Markov process. In order to represent Eqs. (1) as a Markov vector, the acceleration terms Y'', which appear in the nonlinear terms $\psi_i(Y, Y', Y'')$, must be removed by successive elimination. This process has been performed by using the MACSYMA software. Having eliminated Y'', Eqs. (1) can be written in the Stratonovich form[16]

$$X_i' = f_i(X, \tau) + \sum_{j=1}^{4} G_{ij}(X, \tau)\xi_j(\tau), \qquad j = 1, \ldots, 4 \tag{3}$$

through the coordinate transformation

$$\{Y_1, Y_2, Y_1', Y_2'\} = \{X_1, X_2, X_3, X_4\} \tag{4}$$

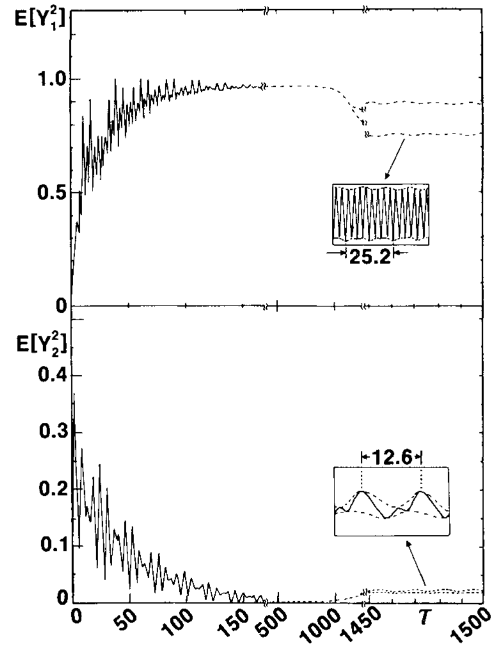

Fig. 2 Time-history response of normal mode mean squares according to Gaussian closure solution.

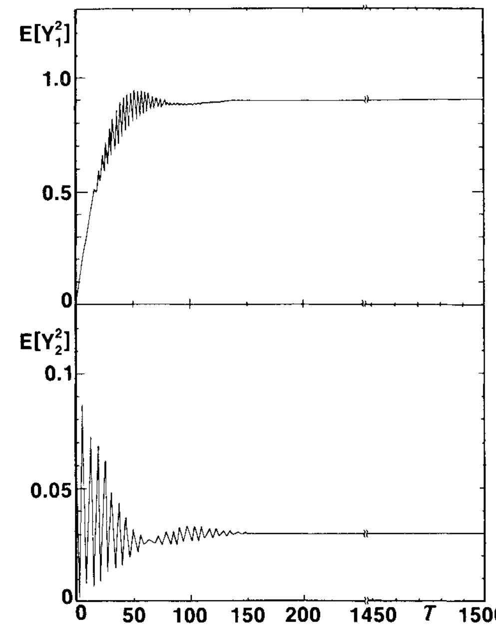

Fig. 3 Time-history response of normal mode mean squares according to non-Gaussian closure solution.

Alternatively, the system equations of motion can be written in terms of the Ito-type equation

$$dX_i = \left[f_i(X, \tau) + \frac{1}{2} \sum_{k=1}^{4} \sum_{j=1}^{4} G_{kj}(X, \tau) \frac{\partial G_{ij}(X, \tau)}{\partial X_k} \right] d\tau + \sum_{j=1}^{4} G_{ij}(X, \tau)\, dB_j(\tau) \tag{5}$$

where the double summation expression is referred to the Ito (or the Wong-Zakai) correction term,[16] which is a result of replacing the physical wide-band random process $\xi_i(\tau)$ by the white noise $W_i(\tau)$. In Eq. (5) the white-noise processes $W_i(\tau)$

have been replaced by the formal derivative of the Brownian motion $B_i(\tau)$, i.e.,

$$W_i(\tau) = \mathrm{d}B_i(\tau)/\mathrm{d}\tau \tag{6}$$

The statistical properties of $B_i(\tau)$ are

$$E[\mathrm{d}B_i(\tau)] = 0 \tag{7a}$$

$$E\left[\mathrm{d}B_i^2(\tau)\right] = 2D_i\,\mathrm{d}\tau \tag{7b}$$

$$E\left[\mathrm{d}B_i(\tau)\,\mathrm{d}B_j(\tau)\right] = 0 \qquad \text{for } i \neq j \tag{7c}$$

where $2D_i$ is the spectral density of the Brownian motion process $B_i(\tau)$. The equations of motion can now be written in terms of the Markov vector coordinates $\boldsymbol{X}$, as

$$
\begin{aligned}
\mathrm{d}X_1 &= X_3\,\mathrm{d}\tau \\
\mathrm{d}X_2 &= X_4\,\mathrm{d}\tau \\
\mathrm{d}X_3 &= \{ -X_1 - 2\zeta_1 X_3 - a_4 X_1^2 - (a_6 + r^2 a_5) X_1 X_2 - r^2 a_7 X_2^2 - 2\zeta_1 a_4 X_1 X_3 - 2\zeta_2 r a_5 X_1 X_4 - 2\zeta_1 a_6 X_2 X_3 - 2\zeta_2 r a_7 X_2 X_4 + a_8 X_3^2 \\
&\quad + a_9 X_3 X_4 + a_{10} X_4^2 \underline{+ 4D_{c_1}\zeta_1^2 (X_3 + 2a_4 X_1 X_3 + 2a_6 X_2 X_3 + a_6^2 X_1^2 X_3 + 2a_4 a_6 X_1 X_2 X_3 + a_6^2 X_2 X_3)} \\
&\quad \underline{+ 4D_{c_2}\zeta_2^2 r^2 [a_5 X_1 X_4 + a_7 X_2 X_4 + a_5 b_5 X_1^2 X_4 + (a_5 b_7 + a_7 b_5) X_1 X_2 X_4 + a_7 b_7 X_2^2 X_4]} \} \,\mathrm{d}\tau - X_1\,\mathrm{d}B_{k_1}(\tau) \\
&\quad - r^2 (a_5 X_1 X_2 + a_7 X_2 X_4)\,\mathrm{d}B_{k_2}(\tau) - 2\zeta_1 (X_3 + a_4 X_1 X_3 + a_6 X_2 X_3)\,\mathrm{d}B_{c_1}(\tau) - 2\zeta_2 r (a_5 X_1 X_4 + a_7 X_2 X_4)\,\mathrm{d}B_{c_2}(\tau) \\
&\quad - (A_1 + A_2 X_1 + A_3 X_2 + A_4 X_1^2 + A_5 X_1 X_2 + A_6 X_2^2)\,\mathrm{d}B_0(\tau) \\
\mathrm{d}X_4 &= \{ -r^2 X_2 - 2\zeta_2 r X_4 - b_4 X_1^2 - (b_6 + r^2 b_5) X_1 X_2 - r^2 b_7 X_2^2 - 2\zeta_1 b_4 X_1 X_3 - 2\zeta_2 r b_5 X_1 X_4 - 2\zeta_1 b_6 X_2 X_3 - 2\zeta_2 r b_7 X_2 X_4 \\
&\quad + b_8 X_3^2 + b_9 X_3 X_4 + b_{10} X_4 + 4D_{c_1}\zeta_1^2 [b_4 X_1 X_3 + b_6 X_2 X_3 \underline{+ a_4 b_4 X_1^2 X_3 + (a_4 b_6 + b_4 a_6) X_1 X_2 X_3 + a_6 b_6 X_2^2 X_3]} \\
&\quad \underline{+ 4D_{c_2}\zeta_2^2 r^2 (X_4 + 2b_5 X_1 X_4 + 2b_7 X_2 X_4 + b_5^2 X_1^2 X_4 + 2b_5 b_7 X_1 X_2 X_4 + b_7^2 X_2^2 X_4)} \} \,\mathrm{d}\tau - 2\zeta_1 (b_4 X_1 X_3 + b_6 X_2 X_3)\,\mathrm{d}B_{c_1}(\tau) \\
&\quad - r^2 (X_2 + b_5 X_1 X_2 + b_7 X_2^2)\,\mathrm{d}B_{k_2}(\tau) - 2\zeta_2 r (X_4 + b_5 X_1 X_4 + b_7 X_2 X_4)\,\mathrm{d}B_{c_2}(\tau) \\
&\quad - (B_1 + B_2 X_1 + B_3 X_2 + B_4 X_1^2 + B_5 X_1 X_2 + B_6 X_2^2)\,\mathrm{d}B_0(\tau)
\end{aligned} \tag{8}
$$

where the underlined expressions are the Wong-Zakai correction terms.

Dynamic Moment Equations

The joint probability density function $p(\boldsymbol{X}, \tau)$ of the response coordinates can be determined by applying the Fokker-Planck equation

$$
\begin{aligned}
\frac{\partial}{\partial \tau} p(\boldsymbol{X}, \tau) &= - \sum_{i=1}^{4} \frac{\partial}{\partial X_i} [a_i(\boldsymbol{X}, \tau) p(\boldsymbol{X}, \tau)] \\
&\quad + \frac{1}{2} \sum_{i=1}^{4} \sum_{j=1}^{4} \frac{\partial^2}{\partial X_i \partial X_j} [b_{ij}(\boldsymbol{X}, \tau) p(\boldsymbol{X}, \tau)]
\end{aligned} \tag{9}
$$

where $a_i(\boldsymbol{X}, \tau)$ and $b_{ij}(\boldsymbol{X}, \tau)$ are the first and second incremental moments of the Markov process $\boldsymbol{X}(\tau)$. These are defined as follows:

$$a_i(\boldsymbol{X}, \tau) = \lim_{\delta\tau \to 0} \frac{1}{\delta\tau} E[X_i(\tau + \delta\tau) - X_i(\tau)] \tag{10a}$$

$$
\begin{aligned}
b_{ij}(\boldsymbol{X}, \tau) &= \lim_{\delta\tau \to 0} \frac{1}{\delta\tau} E \\
&\quad \times \{ [X_i(\tau + \delta\tau) - X_i(\tau)][X_j(\tau + \delta\tau) - X_j(\tau)] \}
\end{aligned} \tag{10b}
$$

provided that all limits exist and $\boldsymbol{X}(\tau) = \boldsymbol{X}$.

It is found that the system Fokker-Planck equation cannot be solved for the response probability density in a closed form. However, it is possible to generate a general differential equation for the response joint moments of any order N by multiplying both sides of the system Fokker-Planck equation by the scaler function

$$\Phi = X_1^i X_2^j X_3^k X_4^\ell \tag{11}$$

where $i + j + k + \ell = N$, and integrating by parts over the entire space $-\infty < X < \infty$. The following notation is adopted to denote the various response moments:

$$m_{i,j,k,\ell} = \iiiint_{-\infty}^{+\infty} X_1^i X_2^j X_3^k X_4 p(X, \tau)\,\mathrm{d}X_1, \mathrm{d}X_2\,\mathrm{d}X_3\,\mathrm{d}X_4 \tag{12}$$

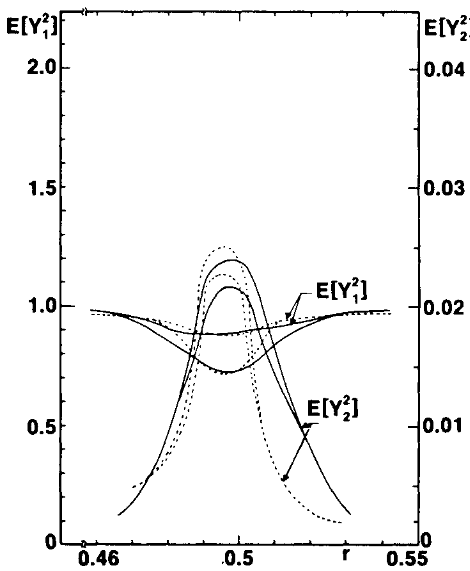

Fig. 4 Mean-square response of normal modes according to Gaussian closure solution vs internal resonance: ——, deterministic system; ---, system with random damping, $D_{c_1} = D_{c_2} = 0.1D_0$, $2D_0 = 0.08$.

The resulting differential equation of the response joint moments is

$$
\begin{aligned}
m'_{i,j,k,\ell} = {} & i m_{i-1,j,k+1,\ell} + j m_{i,j-1,k,\ell+1} + k\Big[-m_{i+1,j,k-1,\ell} + \zeta_1(\zeta_1 D_{c_1} - 2) m_{i,j,k,\ell} - a_4 m_{i+2,j,k-2,\ell} \\
& - (a_6 + r^2 a_5) m_{i+1,j+1,k-1,\ell} - r^2 a_7 m_{i,j+2,k-1,\ell} + 2\zeta_1 a_4(\zeta_1 D_{c_1} - 1) m_{i+1,j,k,\ell} + \zeta_2 r a_5(\zeta_2 r D_{c_2} - 2) m_{i+1,j,k-1,\ell+1} \\
& + 2\zeta_1 a_6(\zeta_1 D_{c_1} - 1) m_{i,j+1,k,\ell} + \zeta_2 r a_7(\zeta_2 r D_{c_2} - 2) m_{i,j+1,k-1,\ell+1} + a_8 m_{i,j,k+1,\ell} + a_9 m_{i,j,k,\ell+1} + a_{10} m_{i,j,k-1,\ell+2}\Big] \\
& + \ell\Big[-r^2 m_{i,j+1,k,\ell-1} + (\zeta_2 r D_{c_2} - 2) m_{i,j,k,\ell} - b_4 m_{i+2,j,k,\ell-1} - (r^2 b_5 + b_6) m_{i+1,j+1,k,\ell-1} - r^2 b_7 m_{i,j+2,k,\ell-1} \\
& + \zeta_1 b_4(\zeta_1 D_{c_1} - 2) m_{i+1,j,k+1,\ell-1} + 2\zeta_2 r b_5(\zeta_2 r D_{c_2} - 1) m_{i+1,j,k,\ell} + \zeta_1 b_6(\zeta_1 D_{c_1} - 2) m_{i,j+1,k+1,\ell-1} \\
& + 2\zeta_2 r b_7(\zeta_2 r D_{c_1} - 1) m_{i,j+1,k,\ell} + b_8 m_{i,j,k+2,\ell-1} + b_9 m_{i,j,k+1,\ell} + b_{10} m_{i,j,k,\ell+1}\Big] + k(k-1)\Big\{D_0 A_1^2 m_{i,j,k-2,\ell} \\
& + 2D_0 A_1 A_2 m_{i+1,j,k-2,\ell} + 2D_0 A_1 A_3 m_{i,j+1,k-2,\ell} + \left[D_{k_1} - D_0(2A_1A_4 + A_2^2)\right] m_{i+1,j,k-2,\ell} \\
& + 2D_0(A_1A_5 + A_2A_3) m_{i+1,j+1,k-2,\ell} + D_0(A_3^2 + 2A_1A_6) m_{i,j+2,k-2,\ell} + 4D_{c_1}\zeta_1^2 m_{i,j,k,\ell}\Big\} + k\ell\Big[2D_0 A_1 B_1 m_{i,j,k+1,\ell+1} \\
& + 2D_0(A_1B_2 + A_2B_1) m_{i+1,j,k-1,\ell-1} + 2D_0(A_1B_3 + A_3B_1) m_{i,j+1,k-1,\ell-1} + 2D_0(A_1B_4 + A_4B_1 + A_2B_2) m_{i+2,j,k-1,\ell-1} \\
& + 2D_0(A_1B_5 + A_5B_1 + A_2B_3 + A_3B_2) m_{i+1,j+1,k-1,\ell-1} + 2D_0(A_3B_3 + A_1B_6 + A_6B_1) m_{i,j+2,k-1,\ell-1}\Big] \\
& + \ell(\ell-1)\Big\{2D_0 B_1^2 m_{i,j,k,\ell-2} + 2D_0 B_1 B_2 m_{i+1,j,k,\ell-2} + 2D_0 B_1 B_3 m_{i,j+1,k,\ell-2} + D_0(2B_1B_4 + B_2^2) m_{i+2,j,k,\ell-2} \\
& + 2D_0(B_1B_5 + B_1B_3) m_{i+1,j+1,k,\ell-2} + \left[D_{k_2} r^2 + D_0(B_3^2 + 2B_1B_6)\right] m_{i,j+2,k,\ell-2} + 4D_{c_2}\zeta_2^2 r^2 m_{i,j,k,\ell}\Big\}
\end{aligned}
\tag{13}
$$

Gaussian Closure Solution

It is seen that any moment equation of order N includes terms of order $N+1$ on the right-hand side of Eq. (13) which, in this case, constitutes an infinite coupled set of moment equations. In order to close this infinite hierarchy, two different closure schemes will be applied. These schemes are based on the properties of the joint cumulant (or semi-invariants). It should be noticed that the response coordinates are not Gaussian-distributed, and any closure scheme should take into account the deviation of the response from being Gaussian. However, if the response coordinates are assumed to be "nearly" Gaussian, then all joint cumulants of order greater than 2 vanish identically, and the response statistics can be described in terms of first- and second-order moments. This approach is referred to as Gaussian closure scheme and is applied for the present system by setting the third-order joint cumulant to zero, i.e.,

$$
\lambda_3[X_iX_jX_k] = E[X_iX_jX_k] - \sum^{3} E[X_i]E[X_jX_k] + 2E[X_i]E[X_j]E[X_k] = 0 \tag{14}
$$

where the number over the summation sign refers to the number of terms generated by the indicated expression without allowing permutation of indices. For example,

$$
\sum^{3} E[X_i]E[X_jX_k] = E[X_i]E[X_jX_k] + E[X_j]E[X_iX_k] + E[X_k]E[X_iX_j] \tag{15}
$$

The Gaussian closure scheme will lead to 14 closed moment equations, which consist of four equations for the first-order moments and ten equations for the second-order moments. These equations will be integrated numerically by using the IMSL (International Mathematical and Statistical Library) DVERK subroutine (Runge-Kutta-Verner fifth- and sixth-order numerical integration method). The results of this solution will be presented in Sec. III.

Non-Gaussian Closure Solution

The Gaussian closure solution is analogous to the linearized solution of nonlinear mechanics problems. In view of the inherent nonlinearity of the system (as well as the random time coefficients), the response processes will be non-Gaussian and, in this case, all higher-order cumulants of order greater than 2 will not vanish. The probability density of non-Gaussian processes can be expressed in terms of the Gram-Charlier expansion or Edgeworth asymptotic series. It has been shown (in Ref. 23) that rapid convergence in the Edgeworth expansion can be achieved by retaining the first few terms in the series. In this paper the non-Gaussian closure solution will be obtained by setting the fifth-order joint cumulant to zero, i.e.,

$$
\begin{aligned}
\lambda_5[X_iX_jX_kX_\ell X_m] = {} & E[X_iX_jX_kX_\ell X_m] \\
& - \sum^{5} E[X_i]E[X_jX_kX_\ell X_m] \\
& + 2\sum^{10} E[X_i]E[X_i]E[X_kX_\ell X_m] \\
& - 6\sum^{10} E[X_i]E[X_j]E[X_k]E[X_\ell X_m] \\
& + 2\sum^{15} E[X_i]E[X_jX_k]E[X_\ell X_m] \\
& - \sum^{10} E[X_iX_j]E[X_kX_\ell X_m] \\
& + 24E[X_i]E[X_j]E[X_k]E[X_\ell]E[X_m]
\end{aligned}
\tag{16}
$$

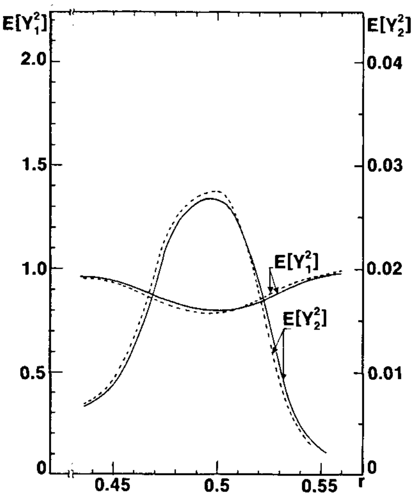

Fig. 5 Mean-square response of normal modes according to non-Gaussian closure solution vs internal resonance: ——, deterministic system; ---, system with random damping, $D_{c_1} = D_{c_2} = 0.1D_0$, $2D_0 = 0.08$.

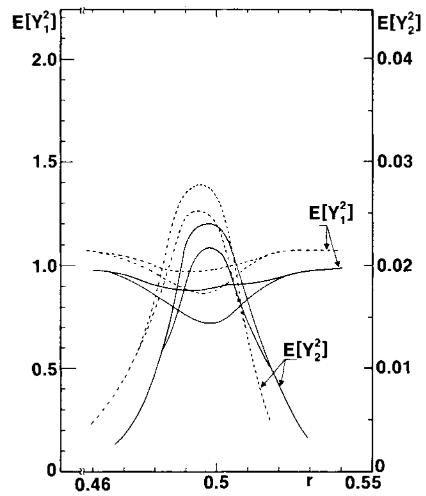

Fig. 6 Mean-square response of normal modes according to Gaussian closure solution vs internal resonance: ——, deterministic system; ---, system with random stiffness, $D_{c_1} = D_{c_2} = 0.1D_0$, $2D_0 = 0.08$.

This procedure requires that 69 differential equations be generated from Eq. (13). These equations consist of 4 equations for first-order moments, 10 equations for second-order moments, 20 equations for third-order moments, and 35 equations for fourth-order moments. These equations will be solved by numerical integration by using the IMSL DVERK subroutine. The results of this solution, together with the Gaussian closure solution, will be discussed in Sec. III.

III. Statistics of the System Response

The statistics of the system response are determined for three different cases of system parameter uncertainties. These are 1) damping random variation, 2) stiffness random variation, and 3) damping and stiffness variations. The results of the numerical integration are presented and discussed in the following sections.

Response of the System with Random Damping

The time-history response of the displacement mean squares of the system normal coordinates is shown in Figs. 2 and 3 according to the Gaussian and non-Gaussian closure solutions, respectively. These figures display both the transient and steady-state responses for exact internal tuning ratio $r = 0.5$, damping ratios $\zeta_1 = \zeta_2 = 0.02$, mass ratio $m_1/m_2 = 0.2$, excitation spectral density $2D_0 = 0.08$, and damping variation density $D_{c_1} = D_{c_2} = 0.1D_0$. Both responses show that the transient response level is greater than the steady-state level. It is seen that after a response period of $\tau = 1000$, the mean squares fluctuate between two limits for the Gaussian closure solution while they are strictly stationary for the non-Gaussian closure solution. This difference between the two solutions is due to the fact that the non-Gaussian closure more adequately models the system nonlinearity. The stationarity of the response of coupled nonlinear systems was verified by Schmidt,[24] who used the stochastic averaging method. The influence of the initial conditions on the time-history response is found to have no effect on the steady-state response for both solutions; however, the effect exists only during the transient period.

The numerical integration has been repeated for various values of internal detuning parameter r in the neighborhood of the exact internal resonance $r = 0.5$. The results are shown in Figs. 4 and 5 for Gaussian and non-Gaussian closure solutions, respectively. These figures include response curves for the case of deterministic system (i.e., the system with constant damping coefficients) for comparison. The Gaussian closure solution is indicated by two curves (for each mode),

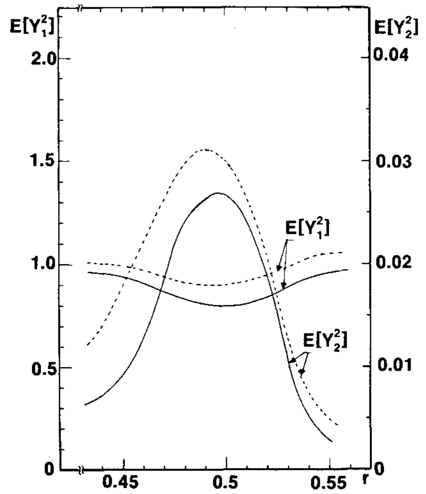

Fig. 7 Mean-square response of normal modes according to non-Gaussian closure solution vs internal resonance: ——, deterministic system; ---, system with random stiffness, $D_{k_1} = D_{k_2} = 0.1D_0$, $2D_0 = 0.08$.

which represent the upper and lower limits of the quasi-stationary response as reflected in the steady-state time-history response shown in Fig. 2. The non-Gaussian solution, on the other hand, is shown by one curve for each mode since the response achieves a stationary response. It is seen that the mean square of the first normal mode approaches the response of the single degree of freedom as the internal detuning is well removed from the exact internal resonance $r = 0.5$. The damping random variation has a remarkable effect on the Gaussian closure solution, and the effect is less pronounced in the non-Gaussian closure solution. In the non-Gaussian solution, the damping variation results in a slight decrease in the mean-square response of the first mode and a corresponding increase in the second mode mean-square response for $r < 0.5$. For $r > 0.5$ the effect is reversed. It is found that the damping fluctuation does not have a uniform effect on the response in the case of Gaussian closure solution; however, the region of the autoparametric interaction becomes narrower.

Response of the System with Random Stiffness

Figures 6 and 7 provide a comparison between the mean-square response of the system as obtained by Gaussian and non-Gaussian closure solutions, respectively. The response of

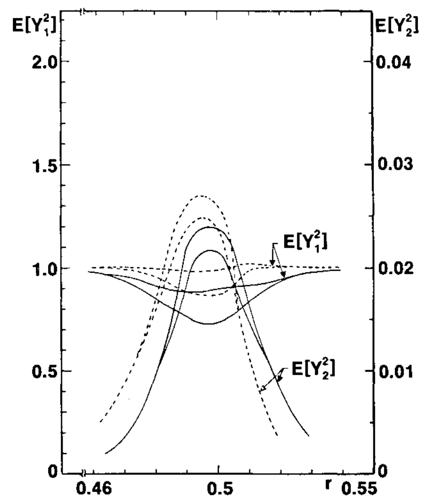

Fig. 8 Mean-square response of normal modes according to Gaussian closure solution vs internal resonance: ——, deterministic system; ---, system with random damping and stiffness, $D_{ci} = D_{ki} = 0.1D_0$, $2D_0 = 0.08$.

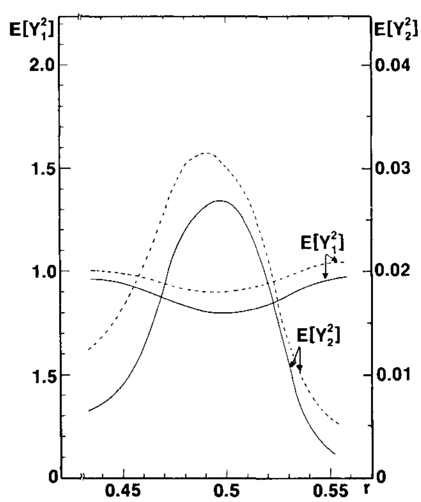

Fig. 9 Mean-square response of normal modes according to non-Gaussian closure solution vs internal resonance: ——, deterministic system; ---, system with random damping and stiffness, $D_{ci} = D_{ki} = 0.1D_0$, $2D_0 = 0.08$.

the system with stiffness random fluctuation is shown by dotted curves, while solid curves belong to the response of the deterministic system. It is seen that, for both solutions, a small random fluctuation in the system stiffness ($D_{k_1} = D_{k_2} = 0.1D_0$) results in a substantial dispersion of the response statistics. It is known that under random stiffness variation the system eigenvalues will be random.[25] Previous investigations[26-29] of the response of linear systems (with parameters represented by random variables) showed that a small dispersion in the stiffness resulted in a considerable dispersion in the system response. It is also evident from both Figs. 6 and 7 that the effect of the stiffness random variation is to increase the level of the response mean squares and to widen the region of autoparametric interaction. However, the characteristics of the autoparametric vibration absorber[30,31] are less effective in the presence of stiffness random variation.

Combined Damping and Stiffness Variations

The system mean-square responses of the system with equal levels of spectral densities of the damping and stiffness random parameters ($D_{ci} = D_{ki} = 0.1D_0$) are shown in Figs. 8 and 9, according to Gaussian and non-Gaussian closure solutions, respectively. Based on the results of the previous two cases, it is clear that the system response is mainly dominated by the stiffness random variation.

IV. Conclusions

The influence of the damping and stiffness random variation on the random response of structural systems with autoparametric interaction has been determined. The Fokker-Planck equation approach has been used to derive a general differential equation for the response moments. This equation constitutes an infinite coupled set of moment equations, which are truncated by two closure schemes. These closure schemes are based on the properties of the statistical cumulants. The first, referred to as Gaussian closure, assumes that the response distribution does not deviate significantly from normal. The other scheme takes into account the deviation of the response distribution from being Gaussian. The results of both solutions are calculated and represented as function of the internal detuning parameter. The Gaussian closure solution results in a quasi-stationary response, while the non-Gaussian closure solution is strictly stationary. It has also been found that the random variation of the system stiffness has more considerable effect than the effect of damping variation on the response mean squares. One last point is that the model selected in this study is a simple structural system in which several characteristics resemble those encountered in aeroelastic structures, such as a wing with external store. However, the investigation did not consider the interaction with random aerodynamic forces that results in stochastic nonlinear flutter. Currently, the authors are involved in a research program supported by the AFOSR to examine the stochastic flutter of real nonlinear models wings and panels under random aerodynamic loading.

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